México ICN-UNAM 97-14

Lie Algebras in Fock Space

by

A. Turbiner*

Instituto de Ciencias Nucleares, UNAM,
Apartado Postal 70-543, 04510 Mexico D.F., Mexico

A catalogue of explicit realizations of representations of Lie (super) algebras and quantum algebras in Fock space is presented.

*On leave of absence from the Institute for Theoretical and Experimental Physics, Moscow 117259, Russia
E-mail: turbiner@axcrnb.cern.ch, turbiner@roxanne.nuclecu.unam.mx

This article is an attempt to present a catalogue of known representations of (super) Lie algebras and quantum algebras acting on different Fock spaces. Of course, we do not have as ambitious goal as to present a complete list of all representations of all possible algebras, but we plan to present some of them, those we consider important for applications, mainly restricting ourselves by those possessing finite-dimensional representations. Many representations are known in the folklore spread throughout the literature under offen different names[1]. Therefore, we provide references according to our taste, knowledge and often quite arbitrarily. This work does not pretend to be totally original. Throughout the text we usually consider complex algebras.

Lie Algebras

1. sl_2-algebra.

Take two operators a and b obeying the commutation relation

$$[a,b] \equiv ab - ba \ = \ 1, \tag{A.1.1}$$

with the identity operator on the r.h.s. – they span the three-dimensional Heisenberg algebra. By definition the universal enveloping algebra of the Heisenberg algebra is the algebra of all normal-ordered polynomials in a, b: any monomial is taken to be of the form $b^k a^m$ [2]. If, besides the polynomials, we also consider holomorphic functions in a, b, the extended universal enveloping algebra of the Heisenberg algebra appears. The Heisenberg-Weil algebra possesses the internal automorphism, which is treated as a certain type of quantum canonical transformations [3]. We say that the (extended) Fock space appears if we take the (extended) universal enveloping algebra of the Heisenberg algebra and add to it the vacuum state $|0>$ such that

$$a|0> = \ 0 \ . \tag{A.1.2}$$

One of the most important realizations of (A.1.1) is the coordinate-momentum representation:

$$a \ = \ \frac{d}{dx} \equiv \partial_x \ , \ b \ = \ x \ , \tag{A.1.3}$$

where $x \in \mathbf{C}$. In this case the vacuum is a constant, say, $|0> = \ 1$. Recently a finite-difference analogue of (A.1.3) has been found [1],

$$a \ = \ \mathcal{D}_+, \ b \ = \ x(1 - \delta\mathcal{D}_-) \ , \tag{A.1.4}$$

where

$$\mathcal{D}_+ f(x) = \frac{f(x+\delta) - f(x)}{\delta} \ ,$$

is the finite-difference operator, $\delta \in C$ and $\mathcal{D}_+ \to \mathcal{D}_-$, if $\delta \to -\delta$.

(a). It is easy to check that if the operators a, b obey (A.1.1), then the following three operators

$$J_n^+ = b^2 a - nb \ ,$$

---

[1] For instance, in nuclear physics some of them are known as boson representations

[2] Sometimes this is called the Heisenberg-Weil algebra

[3] This means that there exists a family of the elements of the Heisenberg-Weil algebra obeying the commutation relation (A.1.1)

$$J_n^0 = ba - \frac{n}{2}\ , \qquad (A.1.5)$$
$$J_n^- = a\ ,$$

span the sl_2-algebra with the commutation relations:

$$[J^0, J^\pm] = \pm J^\pm\ ,\ [J^+, J^-] = -2J^0\ ,$$

where $n \in C$. For the representation (A.1.5) the quadratic Casimir operator is equal to

$$C_2 \equiv \frac{1}{2}\{J^+, J^-\} - J^0 J^0 = -\frac{n}{2}\left(\frac{n}{2} + \frac{1}{2}\right)\ , \qquad (A.1.6)$$

where $\{\ ,\ \}$ denotes the anticommutator. If n is a non-negative integer, then (A.1.5) possesses a finite-dimensional, irreducible representation in the Fock space leaving invariant the space

$$\mathcal{P}_n(b) \ = \ \langle 1, b, b^2, \dots, b^n \rangle, \qquad (A.1.7)$$

of dimension $\dim \mathcal{P}_n = (n+1)$.

Substitution of (A.1.3) into (A.1.5) leads to a well-known representation of sl_2-algebra of differential operators of the first order [4]

$$J_n^+ = x^2 \partial_x - nx\ ,$$
$$J_n^0 = x\partial_x - \frac{n}{2}\ , \qquad (A.1.8)$$
$$J^- = \partial_x\ ,$$

where the finite-dimensional representation space (A.1.7) becomes the space of polynomials of degree not higher than n

$$\mathcal{P}_n(x) \ = \ \langle 1, x, x^2, \dots, x^n \rangle\ . \qquad (A.1.9)$$

(b). The existence of the internal automorphism of the extended universal enveloping algebra of the Heisenberg algebra, i.e $[\hat{a}(a,b), \hat{b}(a,b)] = [a,b] = 1$ allows to construct different representations of the algebra sl_2 by $a \to \hat{a}, b \to \hat{b}$ in (A.1.5). In particular, the internal automorphism of the extended universal enveloping algebra of the Heisenberg algebra is realized by the following two operators,

$$\hat{a} = \frac{(e^{\delta a} - 1)}{\delta}\ ,$$
$$\hat{b} = b e^{-\delta a}\ , \qquad (A.1.10)$$

where δ is any complex number. If δ goes to zero then $\hat{a} \to a, \hat{b} \to b$. In other words, (A.1.10) is a 1-parameter quantum canonical transformation of the deformation type of the Heisenberg algebra (A.1.1). It is the quantum analogue of a point canonical transformation. The substitution of the representation (A.1.10) into (A.1.5) results in the following representation of the sl_2-algebra

$$J_n^+ = (\frac{b}{\delta} - 1) b e^{-\delta a}(1 - n - e^{-\delta a})\ ,$$
$$J_n^0 = \frac{b}{\delta}(1 - e^{-\delta a}) - \frac{n}{2}\ ,\ J^- = \frac{1}{\delta}(e^{\delta a} - 1)\ . \qquad (A.1.11)$$

[4] This representation was known to Sophus Lie.

If n is a non-negative integer, then (A.1.11) possesses a finite-dimensional irreducible representation of the dimension $\dim \mathcal{P}_n = (n+1)$ coinciding with (A.1.7). It is worth noting that the vacuum for (A.1.10) remains the same, for instance (A.1.2). Also the value of the quadratic Casimir operator for (A.1.11) coincides with that given by (A.1.6).

The operator $\hat{a}$ in the particular representation (A.1.4) becomes the well-known translationally-covariant finite-difference operator

$$\hat{a}f(x) = \frac{(e^{\delta\partial_x} - 1)}{\delta} f(x) \ = \ \mathcal{D}_+ f(x) \tag{A.1.12}$$

while $\hat{b}$ takes the form

$$\hat{b}f(x) = xe^{-\delta\partial_x} f(x) \ = \ xf(x-\delta) = x(1-\delta\mathcal{D}_-)f(x) \ . \tag{A.1.13}$$

After substitution of (A.1.12)–(A.1.13) into (A.1.11) we arrive at a representation of the sl_2-algebra by finite-difference operators,

$$J_n^+ = x(\frac{x}{\delta} - 1)e^{-\delta\partial_x}(1 - n - e^{-\delta\partial_x}) \ ,$$

$$J_n^0 = \frac{x}{\delta}(1 - e^{-\delta\partial_x}) - \frac{n}{2} \ , \ J^- = \frac{1}{\delta}(e^{\delta\partial_x} - 1) \ , \tag{A.1.14}$$

or, equivalently,

$$J_n^+ = x(1 - \frac{x}{\delta})(\delta^2\mathcal{D}_-\mathcal{D}_- - (n+1)\delta\mathcal{D}_- + n) \ ,$$

$$J_n^0 = x\mathcal{D}_- - \frac{n}{2} \ , \ J^- = \ \mathcal{D}_+. \tag{A.1.15}$$

The finite-dimensional representation space for (A.1.14)–(A.1.15) for integer values of n is again given by the space (A.1.9) of polynomials of degree not higher than n.

(c). Another example of quantum canonical transformation is given by the oscillatory representation

$$\hat{a} = \frac{b+a}{\sqrt{2}} \ ,$$

$$\hat{b} = \frac{b-a}{\sqrt{2}} \ . \tag{A.1.16}$$

Inserting (A.1.16) into (A.1.5) it is easy to check that the following three generators form a representation of the sl_2-algebra,

$$J_n^+ = \frac{1}{2^{3/2}}[b^3 + a^3 - b(b+a)a - (2n+1)(b-a) - 2b] \ ,$$

$$J_n^0 = \frac{1}{2}(b^2 - a^2 - n - 1) \ , \tag{A.1.17}$$

$$J^- = \frac{b+a}{\sqrt{2}} \ ,$$

where $n \in C$. In this case the vacuum state

$$(b+a)|0> = \ 0, \tag{A.1.18}$$

differs from (A.1.2). If n is a non-negative integer, then (A.1.17) possesses a finite-dimensional irreducible representation in the Fock space

$$\mathcal{P}_n(b) \;=\; \langle 1,(b-a),(b-a)^2,\ldots,(b-a)^n\rangle \;, \qquad (A.1.19)$$

of dimension $\dim\mathcal{P}_n = (n+1)$.

Taking a, b in the realization (A.1.3) and substituting them into (A.1.17), we obtain

$$\begin{gathered} J_n^+ = \frac{1}{2^{3/2}}[x^3+\partial_x^3 - x(x+\partial_x)\partial_x - (2n+1)(x-\partial_x) - 2\partial_x]\ , \\ J_n^0 = \frac{1}{2}(x^2-\partial_x^2-n-1)\ , \qquad (A.1.20) \\ J^- = \frac{x+\partial_x}{\sqrt{2}}\ , \end{gathered}$$

which represents the sl_2-algebra by means of differential operators of finite order (but not of first order as in (A.1.8)). The operator J_n^0 coincides with the Hamiltonian of the harmonic oscillator (with the reference point for eigenvalues changed). The vacuum state is

$$|0> = \; e^{-\frac{x^2}{2}}\ , \qquad (A.1.21)$$

and the representation space is

$$\mathcal{P}_n(x) \;=\; \langle 1,x,x^2,\ldots,x^n\rangle e^{-\frac{x^2}{2}}\ , \qquad (A.1.22)$$

(cf.(A.1.19)).

(d). The following three operators

$$\begin{gathered} J^+ = \frac{a^2}{2}\ , \\ J^0 = -\frac{\{a,b\}}{4}\ , \qquad (A.1.23) \\ J^- = \frac{b^2}{2}\ , \end{gathered}$$

are generators of the sl_2-algebra and the quadratic Casimir operator for this representation is

$$C_2 = \frac{3}{16}\ .$$

This is the so-called metaplectic representation of sl_2 (see, for example, [2]). This representation is infinite-dimensional. Taking the realization (A.1.2) or (A.1.4) of the Heisenberg algebra we get the well-known representation

$$J^+ = \frac{1}{2}\partial_x^2\ ,\ J^0 = -\frac{1}{2}(x\partial_x - \frac{1}{2})\ ,\ J^- = \frac{1}{2}x^2 \qquad (A.1.24)$$

in terms of differential operators, or

$$\begin{gathered} J^+ = \frac{1}{2}\mathcal{D}_+^2\ ,\ J^0 = -\frac{1}{2}(x\mathcal{D}_- - \frac{1}{2})\ , \\ J^- = \frac{1}{2}x(x-\delta)(1-2\delta\mathcal{D}_- - \delta^2\mathcal{D}_-^2)\ , \qquad (A.1.25) \end{gathered}$$

in terms of finite-difference operators, correspondingly.

(e). Take two operators a and b from the Clifford algebra s_2

$$\{a,b\} \equiv ab+ba \ = \ 0 \ , \ a^2=b^2=1 \ . \tag{A.1.26}$$

Then the operators

$$J^1=a \ , \ J^2=b \ , \ J^3=ab \ , \tag{A.1.27}$$

form the sl_2-algebra.

(f). Take the 5-dimensional Heisenberg algebra

$$[a_i,b_j] \ = \ \delta_{ij}, \ i,j=1,2,\dots,p \ , \tag{A.1.28}$$

where δ_{ij} is the Kronecker symbol and $p=2$. The operators

$$J^1=b_1a_2 \ , \ J^2=b_2a_1 \ , \ J^3=b_1a_1-b_2a_2 \ , \tag{A.1.29}$$

form the sl_2-algebra. This representation is reducible. If (A.1.28) is given the coordinate-momentum representation

$$a_i \ = \ \frac{d}{dx_i} \equiv \partial_i \ , \ b_i \ = \ x_i \ , \tag{A.1.30}$$

where $x \in \mathbf{C^2}$, the representation (A.1.29) becomes the well-known vector-field representation. The vacuum is a constant. Finite-dimensional representations appear if a linear space of homogeneous polynomials of fixed degree is taken.

2. sl_3-algebra.

(a). Take the Fock space associated with the 5-dimensional Heisenberg algebra (A.1.28) with vacuum

$$a_i|0> = \ 0 \ , \ i=1,2 \tag{A.2.1}$$

One can show that the following operators are the generators of the sl_3-algebra

$$\begin{gathered}
J_1^+ \ = \ b_1(b_1a_1 \ + \ b_2a_2 - \ n) \ , \ J_2^+ \ = \ b_2(b_1a_1 \ + \ b_2a_2 - \ n) \ , \\
J_1^- \ = \ a_1 \ , \ J_2^- \ = \ a_2 \ , \\
J_{21}^0 \ = \ b_2a_1 \ , \ J_{12}^0 \ = \ b_1a_2 \ , \\
J_1^0 \ = \ b_1a_1-b_2a_2 \ , \ J_2^0 \ = \ b_1a_1+b_2a_2 \ - \ \frac{2}{3}n \ ,
\end{gathered} \tag{A.2.2}$$

where n is a complex number. If n is a non-negative integer, (A.2.2) possesses a finite-dimensional representation and its reprentation space is given by the inhomogeneous polynomials of the degree not higher than n in the Fock space:

$$\mathcal{P}_n = \langle b_1^{n_1} b_2^{n_2} \mid 0 \le (n_1+n_2) \le n\rangle \ . \tag{A.2.3}$$

In the coordinate-momentum representation (A.1.30) the representation (A.2.2) becomes

$$\begin{gathered}
J_1^+ \ = \ x_1(x_1\partial_1 \ + \ x_2\partial_2 - \ n) \ , \ J_2^+ \ = \ x_2(x_1\partial_1 \ + \ x_2\partial_2 - \ n) \ , \\
J_1^- \ = \ \partial_1 \ , \ J_2^- \ = \ \partial_2 \ , \\
J_{21}^0 \ = \ x_2\partial_1 \ , \ J_{12}^0 \ = \ x_1\partial_2 \ , \\
J_1^0 \ = \ x_1\partial_1-x_2\partial_2 \ , \ J_2^0 \ = \ x_1\partial_1+x_2\partial_2 \ - \ \frac{2}{3}n \ ,
\end{gathered} \tag{A.2.4}$$

where the vacuum $|0>= 1$ and for non-negative integer n the space of the finite-dimensional representation is given by

$$\mathcal{P}_n = \langle x_1^{n_1} x_2^{n_2} \mid 0 \le (n_1+n_2) \le n \rangle \ . \tag{A.2.5}$$

(b). An important example of a quantum canonical transformation of the 5-dimensional Heisenberg algebra (A.1.28) is a generalization of (A.1.10) and has the form

$$\hat{a}_i = \frac{(e^{\delta_i a_i}-1)}{\delta_i} \ ,$$
$$\hat{b}_i = b_i e^{-\delta_i a_i} \ , \ i=1,2 \ , \tag{A.2.6}$$

where $\delta_{1,2}$ are complex numbers. Under this transformation the vacuum remains the same (A.2.1). Finally, we are led to the following representation of the sl_3-algebra

$$\begin{aligned} J_1^+ \ &= \ \hat{b}_1(\hat{b}_1\hat{a}_1 \ + \ \hat{b}_2\hat{a}_2 - \ n) \ , \ J_2^+ \ = \ \hat{b}_2(\hat{b}_1\hat{a}_1 \ + \ \hat{b}_2\hat{a}_2 - \ n) \ , \\ J_1^- \ &= \ \hat{a}_1 \ , \ J_2^- \ = \ \hat{a}_2 \ , \\ J_{21}^0 \ &= \ \hat{b}_2\hat{a}_1 \ , \ J_{12}^0 \ = \ \hat{b}_1\hat{a}_2 \ , \\ J_1^0 \ &= \ \hat{b}_1\hat{a}_1 - \hat{b}_2\hat{a}_2 \ , \ J_2^0 \ = \ \hat{b}_1\hat{a}_1 + \hat{b}_2\hat{a}_2 \ - \ \frac{2}{3}n \ , \end{aligned} \tag{A.2.7}$$

As in previous case (a), for a non-negative integer n the representation (A.2.7) becomes finite-dimensional with the corresponding representation space given by (A.2.3). We should mention that in the coordinate-momentum representation the operators $\hat{a}, \hat{b}$ can be rewritten in terms of finite-difference operators (A.1.12-A.1.13), $\mathcal{D}_\pm^{(x,y)}$ and, finally, the generators become

$$\begin{aligned} J_1^+ \ &= \ x(1-\delta_1\mathcal{D}_-^{(x)})(x\mathcal{D}_-^{(x)} \ + \ y\mathcal{D}_-^{(y)} - \ n) \ , \\ J_2^+ \ &= \ y(1-\delta_2\mathcal{D}_-^{(y)})(x\mathcal{D}_-^{(x)} \ + \ y\mathcal{D}_-^{(y)} - \ n) \ , \\ J_1^- \ &= \ \mathcal{D}_+^{(x)} \ , \ J_2^- \ = \ \mathcal{D}_+^{(y)} \ , \\ J_{21}^0 \ &= \ y(1-\delta_2\mathcal{D}_-^{(y)})\mathcal{D}_+^{(x)} \ , \ J_{12}^0 \ = \ x(1-\delta_1\mathcal{D}_-^{(x)})\mathcal{D}_+^{(y)} \ , \\ J_1^0 \ &= \ x\mathcal{D}_-^{(x)} \ - \ y\mathcal{D}_-^{(y)} \ , \ J_2^0 \ = \ x\mathcal{D}_-^{(x)} \ + \ y\mathcal{D}_-^{(y)} \ - \ \frac{2n}{3} \ . \end{aligned} \tag{A.2.8}$$

(c). Another representation of the sl_3-algebra is related to the 7-dimensional Heisenberg algebra (A.1.28) for $p=3$. The generators are

$$\begin{aligned} J_1^+ \ &= \ -(b_2-b_1b_3)a_1 - b_2b_3a_2 - b_3^2a_3 + nb_3 \ , \\ J_2^+ &= -b_1(b_2-b_1b_3)a_1 - b_2^2a_2 - b_2b_3a_3 - mb_1b_3 + (n+m)b_2, \\ J_1^- \ &= \ a_2 \ , \ J_2^- \ = \ a_3 \ , \\ J_{32}^0 \ &= \ a_1 \ + \ b_3a_2 \ , \ J_{23}^0 \ = \ -b_1^2a_1 \ + \ b_2a_3 \ + \ mb_1, \\ J_1^0 \ &= \ -b_1a_1 \ + \ b_2a_2 \ + \ 2b_3a_3 \ - \ n \ , \\ J_2^0 \ &= \ 2b_1a_1 \ + \ b_2a_2 \ - \ b_3a_3 \ - \ m \ , \end{aligned} \tag{A.2.9}$$

where m, n are real numbers. In the coordinate-momentum representation of Heisenberg algebra the algebra (A.2.9) becomes the sl_3-algebra of first order differential operators in the regular representation (on the flag manifold)

$$\begin{aligned} J_1^+ \ &= \ -(y-xz)\partial_x \ - \ yz\partial_y \ - \ z^2\partial_z \ + \ nz \ , \\ J_2^+ &= -x(y-xz)\partial_x - y^2\partial_y - yz\partial_z - mxz + (n+m)y \ , \\ J_1^- \ &= \ \partial_y \ , \ J_2^- \ = \ \partial_z \ , \\ J_{32}^0 \ &= \ \partial_x \ + \ z\partial_y \ , \ J_{23}^0 \ = \ -x^2\partial_x \ + \ y\partial_z \ + \ mx, \\ J_1^0 \ &= \ -x\partial_x \ + \ y\partial_y \ + \ 2z\partial_z \ - \ n \ , \\ J_2^0 \ &= \ 2x\partial_x \ + \ y\partial_y \ - \ z\partial_z \ - \ m \ . \end{aligned} \tag{A.2.10}$$

Using the realization (A.2.6) of the generators of the Heisenberg algebra H_7 and the coordinate-momentum representation, a realization of the sl_3-algebra emerges in terms of finite-difference operators acting on C^3 functions, which is similar to (A.2.8).

3. $gl_2 \ltimes \mathbf{C}^{r+1}$-algebra

Among the subalgebras of the (extended) universal enveloping algebra of the Heisenberg algebra H_5 there is the 1-parameter family of non-semi-simple algebras $gl_2 \ltimes \mathbf{C}^{r+1}$:

$$\begin{aligned} J^1 \ &= \ a_1 \ , \\ J^2 \ &= \ b_1 a_1 \ - \ \frac{n}{3} \ , \ J^3 \ = \ b_2 a_2 \ - \ \frac{n}{3r} \ , \\ J^4 \ &= \ b_1^2 a_1 \ + \ r b_1 b_2 a_2 \ - \ n b_1 \ , \\ J^{5+k} \ &= \ b_1^k a_2 \ , \ k = 0, 1, \ldots, r \ , \end{aligned} \tag{A.3.1}$$

where $r = 1, 2, \ldots$ and n is a complex number. Here the generators J^{5+k}, $k = 0, 1, \ldots, r$ span the $(r+1)$-dimensional abelian subalgebra $\mathbf{C}^{r+1}$. If n is a non-negative integer, the finite-dimensional representation in the corresponding Fock space occurs,

$$\mathcal{P}_n = \langle b_1^{n_1} b_2^{n_2} \mid 0 \leq (n_1 + r n_2) \leq n \rangle \ . \tag{A.3.2}$$

Taking the concrete realization of the Heisenberg algebra in terms of differential or finite-difference operators in two variables similar to (A.1.2) or (A.1.4) respectively, in the generators (A.3.1) we arrive at the $gl_2 \ltimes \mathbf{C}^{r+1}$-algebra realized as the algebra of first-order differential operators [5] or finite-difference operators, respectively.

[5] This algebra acting on functions of two complex variables realized by vector fields was found by Sophus Lie and, recently, it has been extended to the algebra of first order differential operators [3].

4. gl_k-algebra.

The minimal Fock space where the gl_k-algebra acts is associated with the $(2k-1)$-dimensional Heisenberg algebra H_{2k-1}. The explicit formulas for the generators are given by

$$
\begin{aligned}
J_i^- &= a_i\ , \quad i=2,3,\ldots,k\ ,\\
J_{i,j}^0 &= b_i J_j^- = b_i a_j\ , \quad i,j=2,3,\ldots,k\ ,\\
J^0 &= n-\sum_{p=2}^{k} b_p a_p\ ,\\
J_i^+ &= b_i J^0\ , \quad i=2,3,\ldots,k\ ,
\end{aligned} \tag{A.4.1}
$$

where the parameter n is a complex number. The generators $J_{i,j}^0$ span the algebra gl_{k-1}. If n is a non-negative integer, the representation (A.4.1) becomes the finite-dimensional representation acting on the space of polynomials

$$
V_n(t) \ = \ \mathrm{span}\{b_2^{n_2} b_3^{n_3} b_4^{n_4}\ldots b_k^{n_k}\ |\ 0\le \sum n_i \le n\}\ . \tag{A.4.2}
$$

Substituting the a,b-generators of the Heisenberg algebra in the coordinate-momentum representation into (A.4.1) and using the vacuum, $|0>=1$, we get a representation of the gl_k-algebra in terms of first-order differential operators (see, for example, [4])

$$
\begin{aligned}
J_i^- &= \frac{\partial}{\partial x_i}\ , \quad i=2,3,\ldots,k\ ,\\
J_{i,j}^0 &= x_i J_j^- = x_i\frac{\partial}{\partial x_j}\ , \quad i,j=2,3,\ldots,k\ ,\\
J^0 &= n-\sum_{p=2}^{k} x_p\frac{\partial}{\partial x_p}\ ,\\
J_i^+ &= x_i J^0\ , \quad i=2,3,\ldots,k\ ,
\end{aligned} \tag{A.4.3}
$$

which acts on functions of $x\in \mathbf{C^{k-1}}$. One of the generators, namely $J^0+\sum_{p=2}^{k} J_{p,p}^0$ is proportional to a constant and, if it is taken out, we end up with the sl_k-subalgebra of the original algebra. The generators $J_{i,j}^0$ form the sl_{k-1}-algebra of the vector fields. If n is a non-negative integer, the representation (A.4.3) becomes the finite-dimensional representation acting on the space of polynomials

$$
V_n(x) \ = \ \mathrm{span}\{x_2^{n_2} x_3^{n_3} x_4^{n_4}\ldots x_k^{n_k}\ |\ 0\le \sum n_i \le n\}\ . \tag{A.4.4}
$$

This representation corresponds to a Young tableau of one row with n blocks and is irreducible.

If the a,b-generators of the Heisenberg algebra are taken in the form of finite-difference operators (A.1.4) and are inserted into (A.4.1), the gl_k-algebra appears as the algebra of the finite-difference operators:

$$
\begin{aligned}
J_i^- &= \mathcal{D}_+^{(i)}\ , \quad i=2,3,\ldots,k\ ,\\
J_{i,j}^0 &= x_i(1-\delta_i\mathcal{D}_-^{(i)})J_j^- = x_i(1-\delta_i\mathcal{D}_-^{(i)})\mathcal{D}_+^{(j)},\ i,j=2,3,\ldots k,
\end{aligned}
$$

$$J^0 = n - \sum_{p=2}^{k} x_p \mathcal{D}_-^{(p)} \ ,$$

$$J_i^+ = x_i(1 - \delta_i \mathcal{D}_-^{(i)}) J^0 \ , \quad i = 2, 3, \dots, k \ , \tag{A.4.5}$$

where $\mathcal{D}_\pm^{(i)}$ denote the finite-difference operators (cf.(A.1.4)) acting in the direction x_i.

Lie Super-Algebras

In order to work with superalgebras we must introduce the super Heisenberg algebra. This is the $(2k+2r+1)$-dimensional algebra which contains the H_{2k+1}-Heisenberg algebra (A.1.28) as a subalgebra and also the Clifford algebra s_r :

$$\{a_i^{(f)}, a_j^{(f)}\} \ = \ \{b_i^{(f)}, b_j^{(f)}\} \ = \ 0 \ ,$$

$$\{a_i^{(f)}, b_j^{(f)}\} \ = \ \delta_{ij} \ , \ i, j = 1, 2, \dots, r \ , \tag{S.1}$$

as another subalgebra. There are two widely used realizations of the Clifford algebra (S.1):

(i) The fermionic analogue of the coordinate-momentum representation (A.1.30):

$$a_i^{(f)} = \theta_i^+ \ , \ b_i^{(f)} \ = \ \theta_i \ , \ i = 1, 2, \dots, r \ , \tag{S.2}$$

or, differently,

$$a_i^{(f)} = \frac{\partial}{\partial \theta_i} \ , \ b_i^{(f)} \ = \ \theta_i \ , \ i = 1, 2, \dots, r \ , \tag{S.3}$$

and

(ii) The matrix representation

$$a_i^{(f)} \ = \ \underbrace{\sigma^0 \otimes \ldots \otimes \sigma^0}_{i-1} \otimes \ \sigma^+ \otimes \underbrace{\mathbf{1} \otimes \ldots \otimes \mathbf{1}}_{r-i} \ ,$$

$$b_i^{(f)} \ = \ \underbrace{\sigma^0 \otimes \ldots \otimes \sigma^0}_{i-1} \otimes \ \sigma^- \otimes \underbrace{\mathbf{1} \otimes \ldots \otimes \mathbf{1}}_{r-i} \ , \quad i = 1, 2, \dots, r \ , \tag{S.4}$$

where the $\sigma^{\pm,0}$ are Pauli matrices in standard notation,

$$\sigma^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \ , \ \sigma^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \ , \ \sigma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ .$$

In what follows we will consider the Fock space and also the realizations of the superalgebras assuming that the Clifford algebra generators are taken in the fermionic representation (S.3) or the matrix representation (S.4).

1. $osp(2,2)$-algebra.

Let us define a spinorial Fock space as a linear space of all 2-component spinors with normal ordered polynomials in a, b as components and with a definition of the vacuum

$$|0> = \ \begin{pmatrix} |0>_1 \\ |0>_2 \end{pmatrix}$$

such that any component is annihilated by the operator a:

$$a|0>_i = 0 \ , \ i = 1,2 \tag{S.1.1}$$

(a). Take the Heisenberg algebra (A.1.1). Then consider the following two sets of 2×2 matrix operators:

$$\begin{aligned} T^+ &= b^2 a - nb + b\sigma^-\sigma^+ , \\ T^0 &= ba - \frac{n}{2} + \frac{1}{2}\sigma^-\sigma^+ \ , \\ T^- &= a \ , \\ J &= -\frac{n}{2} - \frac{1}{2}\sigma^-\sigma^+ \ , \end{aligned} \tag{S.1.2}$$

called bosonic (even) generators and

$$Q \ = \ \begin{bmatrix} \sigma^+ \\ b\sigma^+ \end{bmatrix} \ , \ \bar{Q} \ = \ \begin{bmatrix} (ba-n)\sigma^- \\ -a\sigma^- \end{bmatrix} \ , \tag{S.1.3}$$

called fermionic (odd) generators. The explicit matrix form of the even generators is given by:

$$T^+ = \begin{pmatrix} J_n^+ & 0 \\ 0 & J_{n-1}^+ \end{pmatrix}, T^0 = \begin{pmatrix} J_n^0 & 0 \\ 0 & J_{n-1}^0 \end{pmatrix}, T^- = \begin{pmatrix} J^- & 0 \\ 0 & J^- \end{pmatrix},$$
$$J = \begin{pmatrix} -\frac{n}{2} & 0 \\ 0 & -\frac{n+1}{2} \end{pmatrix} \ ,$$

and of the odd ones by

$$Q_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \ , \ Q_2 = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \ ,$$
$$\overline{Q}_1 = \begin{pmatrix} 0 & 0 \\ ba-n & 0 \end{pmatrix} \ , \ \overline{Q}_2 = \begin{pmatrix} 0 & 0 \\ -a & 0 \end{pmatrix} \ , \tag{S.1.4}$$

where the $J_n^{\pm,0}$ are the generators of sl_2 given by (A.1.10).

The above generators span the superalgebra $osp(2,2)$ with the commutation relations:

$$\begin{gathered} [T^0, T^\pm] = \pm T^\pm \quad , \quad [T^+, T^-] = -2T^0 \quad , \quad [J, T^\alpha] = 0 \quad , \alpha = \pm, 0 \\ \{Q_1, \overline{Q}_2\} = -T^- \quad , \quad \{Q_2, \overline{Q}_1\} = T^+ \ , \\ \frac{1}{2}(\{\overline{Q}_1, Q_1\} + \{\overline{Q}_2, Q_2\}) = J \ , \ \frac{1}{2}(\{\overline{Q}_1, Q_1\} - \{\overline{Q}_2, Q_2\}) = T^0 \ , \\ \{Q_1, Q_1\} = \{Q_2, Q_2\} = \{Q_1, Q_2\} = 0 \ , \\ \{\overline{Q}_1, \overline{Q}_1\} = \{\overline{Q}_2, \overline{Q}_2\} = \{\overline{Q}_1, \overline{Q}_2\} = 0 \ , \\ [Q_1, T^+] = Q_2 \ , \ [Q_2, T^+] = 0 \ , \ [Q_1, T^-] = 0 \ , \ [Q_2, T^-] = -Q_1 \ , \\ [\overline{Q}_1, T^+] = 0 \ , \ [\overline{Q}_2, T^+] = -\overline{Q}_1 \ , \ [\overline{Q}_1, T^-] = \overline{Q}_2 \ , \ [\overline{Q}_2, T^-] = 0 \ , \\ [Q_{1,2}, T^0] = \pm\frac{1}{2}Q_{1,2} \quad , \quad [\overline{Q}_{1,2}, T^0] = \mp\frac{1}{2}\overline{Q}_{1,2} \ , \\ [Q_{1,2}, J] = -\frac{1}{2}Q_{1,2} \quad , \quad [\overline{Q}_{1,2}, J] = \frac{1}{2}\overline{Q}_{1,2} \ . \end{gathered} \tag{S.1.5}$$

If, in the expressions (S.1.2)–(S.1.3), the parameter n is a non-negative integer , then (S.1.2)–(S.1.3) possess a finite-dimensional representation in the spinorial Fock space

$$\mathcal{P}_{n,n-1} \;=\; \left\langle \begin{array}{c} 1, b, b^2, \ldots, b^n \\ 1, b, b^2, \ldots, b^{n-1} \end{array} \right\rangle = \left\langle \begin{array}{c} \mathcal{P}_n \\ \mathcal{P}_{n-1} \end{array} \right\rangle . \tag{S.1.6}$$

If we take a representation (A.1.3) of the Heisenberg algebra, the generators (S.1.2) become 2×2 matrix differential operators, where the bosonic generators are [5]

$$\begin{aligned} T^+ &= x^2\partial_x - nx + x\sigma^-\sigma^+ , \\ T^0 &= x\partial_x - \frac{n}{2} + \frac{1}{2}\sigma^-\sigma^+ \;, \\ T^- &= \partial_x \;, \\ J &= -\frac{n}{2} - \frac{1}{2}\sigma^-\sigma^+ \;, \end{aligned} \tag{S.1.7}$$

and the fermionic generators

$$Q \;=\; \begin{bmatrix} \sigma^+ \\ x\sigma^+ \end{bmatrix} , \; \bar{Q} \;=\; \begin{bmatrix} (x\partial_x - n)\sigma^- \\ -\partial_x\sigma^- \end{bmatrix} . \tag{S.1.8}$$

The finite-dimensional representation space for non-negative integer values of the parameter n in (S.1.7-S.1.8) becomes a linear space of 2-component spinors with polynomial components:

$$\mathcal{P}_{n,n-1}(x) \;=\; \left\langle \begin{array}{c} 1, x, x^2, \ldots, x^n \\ 1, x, x^2, \ldots, x^{n-1} \end{array} \right\rangle = \left\langle \begin{array}{c} \mathcal{P}_n(x) \\ \mathcal{P}_{n-1}(x) \end{array} \right\rangle \tag{S.1.9}$$

(b). Taking the quantum canonical transformation (A.1.9) and substituting it into (S.1.2) we arrive at the $osp(2,2)$-algebra analogue of the representation (A.1.11) for the sl_2-algebra,

$$\begin{aligned} T^+ &= (\frac{b}{\delta} - 1)be^{-\delta a}(1 - n - e^{-\delta a} + \sigma^-\sigma^+) \;, \\ T^0 &= \frac{b}{\delta}(1 - e^{-\delta a}) - \frac{n}{2} + \frac{\sigma^-\sigma^+}{2} \;, \\ T^- &= \frac{1}{\delta}(e^{\delta a} - 1) \;, \\ J &= -\frac{1}{2} - \frac{\sigma^-\sigma^+}{2} \;, \end{aligned} \tag{S.1.10}$$

and,

$$Q \;=\; \begin{bmatrix} \sigma^+ \\ be^{-\delta a}\sigma^+ \end{bmatrix} , \; \bar{Q} \;=\; \begin{bmatrix} \frac{b - be^{-\delta a} - n}{\delta}\sigma^- \\ \frac{1 - e^{\delta a}}{\delta}\sigma^- \end{bmatrix} . \tag{S.1.11}$$

Taking for the generators a, b the coordinate-momentum realization (A.1.3), we obtain a representation of the algebra $osp(2,2)$ in terms of finite-difference operators

$$\begin{aligned} T^+ &= (\frac{x}{\delta} - 1)xe^{-\delta\partial_x}(1 - n - e^{-\delta\partial_x} + \sigma^-\sigma^+) \;, \\ T^0 &= \frac{x}{\delta}(1 - e^{-\delta\partial_x}) - \frac{n}{2} + \frac{\sigma^-\sigma^+}{2} \;, \end{aligned} \tag{S.1.12}$$

$$T^- = \frac{1}{\delta}(e^{\delta\partial_x} - 1) \ ,$$
$$J = -\frac{1}{2} - \frac{\sigma^-\sigma^+}{2} \ ,$$

and,

$$Q \ = \ \begin{bmatrix} \sigma^+ \\ xe^{-\delta\partial_x}\sigma^+ \end{bmatrix} \ , \ \bar{Q} \ = \ \begin{bmatrix} \frac{x - xe^{-\delta\partial_x} - n}{\delta}\sigma^- \\ \frac{1-e^{\delta\partial_x}}{\delta}\sigma^- \end{bmatrix} \ . \tag{S.1.13}$$

Or, in terms of the operators $\mathcal{D}_\pm$, their explicit matrix forms are the following

$$T^+ = \begin{pmatrix} J_n^+ & 0 \\ 0 & J_{n-1}^+ \end{pmatrix}, T^0 = \begin{pmatrix} x\mathcal{D}_- - \frac{n}{2} & 0 \\ 0 & x\mathcal{D}_- - \frac{n-1}{2} \end{pmatrix}, T^- = \begin{pmatrix} \mathcal{D}_+ & 0 \\ 0 & \mathcal{D}_+ \end{pmatrix},$$
$$J = \begin{pmatrix} -\frac{n}{2} & 0 \\ 0 & -\frac{n+1}{2} \end{pmatrix} \ ,$$

for the bosonic generators and

$$Q_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \ , \ Q_2 = \begin{pmatrix} 0 & x(1-\delta\mathcal{D}_-) \\ 0 & 0 \end{pmatrix} \ ,$$
$$\overline{Q}_1 = \begin{pmatrix} 0 & 0 \\ x\mathcal{D}_- - n & 0 \end{pmatrix} \ , \ \overline{Q}_2 = \begin{pmatrix} 0 & 0 \\ -\mathcal{D}_+ & 0 \end{pmatrix} \ , \tag{S.1.14}$$

for the fermionic generators, where the generator J_n^+ is given by (A.1.15).

(c). The super-metaplectic representation of the $osp(2,2)$-algebra can be easily constructed and has the following form. The even generators are given by

$$T^+ = \begin{pmatrix} \frac{a^2}{2} & 0 \\ 0 & \frac{a^2}{2} \end{pmatrix}, T^0 = \begin{pmatrix} -\frac{\{a,b\}}{4} & 0 \\ 0 & -\frac{\{a,b\}}{4} \end{pmatrix}, T^- = \begin{pmatrix} \frac{b^2}{2} & 0 \\ 0 & \frac{b^2}{2} \end{pmatrix},$$
$$J \ = \ \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & -\frac{1}{4} \end{pmatrix} \ ,$$

while the odd ones are

$$Q_1 = \begin{pmatrix} 0 & -\frac{b}{\sqrt{2}} \\ 0 & 0 \end{pmatrix} \ , \ Q_2 = \begin{pmatrix} 0 & \frac{a}{\sqrt{2}} \\ 0 & 0 \end{pmatrix} \ ,$$
$$\overline{Q}_1 = \begin{pmatrix} 0 & 0 \\ \frac{a}{\sqrt{2}} & 0 \end{pmatrix} \ , \ \overline{Q}_2 = \begin{pmatrix} 0 & 0 \\ \frac{b}{\sqrt{2}} & 0 \end{pmatrix} \ . \tag{S.1.15}$$

Taking the realization of the Heisenberg algebra H_3 in terms of the differential or finite-difference operators (A.1.2), (A.1.4), respectively, and inserting it into (S.1.15) we end up with a realization of the super-metaplectic representation of the $osp(2,2)$-algebra in terms of differential or finite-difference operators.

2. $gl(k+1, r+1)$-superalgebra.

One of the simplest representations of the $gl(k+1, r+1)$-superalgebra can be written as follows

$$T_i^- = a_i \ , \quad i = 1, 2, \dots, k \ ,$$
$$T_{i,j}^0 = b_i T_j^- = b_i a_j \ , \quad i, j = 1, 2, \dots, k \ ,$$

$$
\begin{aligned}
T^0 &= n - \sum_{p=1}^{k} b_p a_p - \sum_{p=1}^{r} \theta_p \frac{\partial}{\partial \theta_p} \ , \\
T_i^+ &= b_i T^0 \ , \quad i = 1,2,\dots,k \ , \\
\overline{Q}_i^- &= \frac{\partial}{\partial \theta_i} \ , \quad i = 1,2,\dots,r \ , \\
\overline{Q}_i^+ &= \theta_i T^0 \ , \quad i = 1,2,\dots,r \ , \\
Q_{ij}^- &= \theta_i T_j^- = \theta_i a_j \ , \quad i,j = 1,2,\dots,r \ , \\
Q_{ij}^+ &= b_i \overline{Q}_j^- = b_i \frac{\partial}{\partial \theta_i} \ , \quad i = 1,2,\dots,k \ , \ j = 1,2,\dots,r \ , \\
J_{i,j}^0 &= \theta_i \overline{Q}_i^- = \theta_i \frac{\partial}{\partial \theta_j} \ , \quad i,j = 1,2,\dots,r \ ,
\end{aligned}
\tag{S.2.1}
$$

These generators can be represented by the following $(k+p)\times(k+p)$ matrix,

$$
\left(\begin{array}{ccc}
k\times k & | & p\times k \\
BB & | & BF \\
----- & & ---- \\
k\times p & | & p\times p \\
FB & | & FF
\end{array}\right) \ ,
\tag{S.2.2}
$$

where the notation $B(F)B(F)$ means the product of a bosonic operator B (fermionic F) with a bosonic operator B (fermionic F). Correspondingly, the operators T in (S.2.1) are of BB-type (mixed with FF-type), J are of FF-type, while the rest operators are of BF-type. The algebra is defined by the (anti)commutation relations

$$
\{[E_{IJ}, E_{KL}]\} = \delta_{IL} E_{JK} \pm \delta_{JK} E_{IL} \ ,
$$

where the generalized indices $I, J, K, L \ = \ B, F$. Anticommutators are taken for generators of FB, BF types only, while for all other cases the defining relations are given by commutators. The dimension of the algebra is $(k+p)^2$.

The generators $J_{i,j}^0$ span the sl_k-algebra of the vector fields. The parameter n in (S.2.1) can be any complex number. However, if n is a non-negative integer, the representation (S.2.1) becomes the finite-dimensional representation acting on a subspace of the Fock space

$$
V_n(b) = \mathrm{span}\{b_1^{n_1} b_2^{n_2} b_3^{n_3} \dots b_k^{n_k} \theta_1^{m_1} \theta_2^{m_2} \dots \theta_r^{m_r} | 0 \le \sum n_i + \sum m_j \le n\}.
\tag{S.2.3}
$$

Taking the coordinate-momentum realization of the Heisenberg algebra (A.1.30) in the generators (S.2.1), we obtain the $gl(k+1, r+1)$-superalgebra realized in terms of first order differential operators (see, for example, [4]):

$$
\begin{aligned}
T_i^- &= \frac{\partial}{\partial x_i} \ , \quad i = 1,2,\dots,k \ , \\
T_{i,j}^0 &= x_i T_j^- = x_i \frac{\partial}{\partial x_j} \ , \quad i,j = 1,2,\dots,k \ , \\
T^0 &= n - \sum_{p=1}^{k} x_p \frac{\partial}{\partial x_p} - \sum_{p=1}^{r} \theta_p \frac{\partial}{\partial \theta_p} \ ,
\end{aligned}
$$

$$
\begin{aligned}
T_i^+ &= x_i T^0 \ , \quad i=1,2,\dots,k \ , \\
\overline{Q}_i^- &= \frac{\partial}{\partial \theta_i} \ , \quad i=1,2,\dots,r \ , \\
\overline{Q}_i^+ &= \theta_i T^0 \ , \quad i=1,2,\dots,r \ , \\
Q_{ij}^- = \theta_i T_j^- &= \theta_i \frac{\partial}{\partial x_j} \ , \quad i=1,2,\dots,r; j=1,2,\dots,k \ , \\
Q_{ij}^+ = x_i \overline{Q}_j^- &= x_i \frac{\partial}{\partial \theta_i} \ , \quad i=1,2,\dots,k; j=1,2,\dots,r \ , \\
J_{i,j}^0 = \theta_i \overline{Q}_i^- &= \theta_i \frac{\partial}{\partial \theta_j} \ , \quad i,j=1,2,\dots,r \ ,
\end{aligned}
\tag{S.2.4}
$$

which acts on functions in $\mathbf{C}^k \otimes \mathbf{G}^r$.

A combination of the generators $J^0 + \sum_{p=1}^k T_{p,p}^0 + \sum_{p=1}^r J_{p,p}^0$, is proportional to a constant and, if it is taken out, we end up with the superalgebra $sl(k+1, r+1)$. The generators $T_{i,j}^0,\ J_{p,q}^0,\ i,j=1,2,\dots,k\ ,\ p,q=1,2,\dots,r$ span the algebra of the vector fields $gl(k,r)$. The parameter n in (S.2.4) can be any complex number. If n is a non-negative integer, the representation (S.2.1) becomes the finite-dimensional representation acting on the space of polynomials

$$
V_n(t) = \mathrm{span}\{x_1^{n_1} x_2^{n_2} x_3^{n_3} \dots x_k^{n_k} \theta_1^{m_1} \theta_2^{m_2} \dots \theta_r^{m_r} | 0 \le \sum n_i + \sum m_j \le n\}.
\tag{S.2.5}
$$

This representation corresponds to a Young tableau of one row with n blocks in the bosonic direction and is irreducible.

If the a, b-generators of the Heisenberg algebra are taken in the form of finite-difference operators (A.1.4) and are inserted into (A.4.1), the $gl(k+1, r+1)$-algebra appears as the algebra of the finite-difference operators:

$$
\begin{aligned}
T_i^- &= \mathcal{D}_+^{(i)} \ , \quad i=1,2,\dots,k \ , \\
T_{i,j}^0 = x_i(1-\delta_i \mathcal{D}_-^{(i)}) T_j^- &= x_i(1-\delta_i \mathcal{D}_-^{(i)}) \mathcal{D}_+^{(j)} \ , \quad i,j=1,2,\dots,k \ , \\
T^0 &= n - \sum_{p=1}^k x_p \mathcal{D}_-^{(p)} - \sum_{p=1}^r \theta_p \frac{\partial}{\partial \theta_p} \ , \\
T_i^+ &= x_i(1-\delta_i \mathcal{D}_-^{(i)}) T^0 \ , \quad i=1,2,\dots,k \ , \\
\overline{Q}_j^- &= \frac{\partial}{\partial \theta_j} \ , \quad j=1,2,\dots,r \ , \\
\overline{Q}_j^+ &= \theta_j T^0 \ , \quad j=1,2,\dots,r \ , \\
Q_{ij}^- = \theta_i T_j^- &= \theta_i \mathcal{D}_+^{(j)} \ , \quad i=1,2,\dots,r; j=1,2,\dots,k \ , \\
Q_{ij}^+ = x_i(1-\delta_i \mathcal{D}_-^{(i)}) \overline{Q}_j^- &= x_i(1-\delta_i \mathcal{D}_-^{(i)}) \frac{\partial}{\partial \theta_i}, \ i=1,2,\dots k; j=1,2,\dots r, \\
J_{i,j}^0 = \theta_i \overline{Q}_i^- &= \theta_i \frac{\partial}{\partial \theta_j} \ , \quad i,j=1,2,\dots r \ ,
\end{aligned}
\tag{S.2.6}
$$

It is worth mentioning that for the integer n, the algebra (S.2.6) has the same finite-dimensional representation (S.2.5) as the algebra of the first order differential operators (S.2.4).

Quantum Algebras

sl_{2q}-algebra.

Take two operators $\tilde{a}$ and $\tilde{b}$ obeying the commutation relation

$$\tilde{a}\tilde{b} - q\tilde{b}\tilde{a} \ = \ 1 \ , \qquad \text{(Q.1)}$$

with the identity operator on the r.h.s. They define the so-called q-deformed Heisenberg algebra. Here $q \in C$. One can define a q-deformed analogue of the universal enveloping algebra by taking all ordered monomials $\tilde{b}^k\tilde{a}^m$. Introducing a vacuum

$$\tilde{a}|0> = \ 0 \ , \qquad \text{(Q.2)}$$

in addition to the q-deformed analogue of the universal enveloping algebra we arrive at a construction which is a q-analogue of Fock space.

It can be easily checked that the q-deformed Heisenberg algebra is a subalgebra of the extended universal enveloping Heisenberg algebra. This can be shown explicitly as follows. For any $q \in C$, two elements of the extended universal enveloping Heisenberg algebra

$$\tilde{a} \ = \ \left(\frac{1}{b}\right)\left(\frac{q^{ba}-1}{q-1}\right) \ , \ \tilde{b} \ = \ b \ , \qquad \text{(Q.3)}$$

obey the commutation relations (Q.1). It can be shown that the universal enveloping Heisenberg algebra does not contain the q-deformed Heisenberg algebra as a subalgebra. The formula (Q.3) allows us to construct different realizations of the the q-deformed Heisenberg algebra. One of them is a q-analogue of the coordinate-momentum representation (A.1.3):

$$\tilde{a} \ = \ \tilde{D}_x \ , \ \tilde{b} \ = \ x \ , \qquad \text{(Q.4)}$$

where

$$\tilde{D}_x f(x) \ = \ \frac{f(qx)-f(x)}{x(q-1)} \ , \qquad \text{(Q.5)}$$

is the so-called Jackson symbol or the Jackson derivative.

Another realization of the (Q.1) appears if a quantum canonical transformation of the Heisenberg algebra (A.1.10) is taken:

$$\tilde{a} \ = \ \left(\frac{1}{b+\delta}\right)e^{\delta a}\left(\frac{q^{\frac{b}{\delta}(1-e^{-\delta a})}-1}{q-1}\right) \ , \ \tilde{b} \ = \ be^{-\delta a} \ , \qquad \text{(Q.6)}$$

where δ is any complex number. In terms of translationally-covariant finite-difference operators $\mathcal{D}_\pm$ the realization has the form

$$\tilde{a} \ = \ \left(\frac{1}{x+\delta}\right)(\delta\mathcal{D}_+ + 1)\left(\frac{q^{x\mathcal{D}_-}-1}{q-1}\right) \ , \ \tilde{b} \ = \ x(1-\delta\mathcal{D}_-) \ . \qquad \text{(Q.7)}$$

In these cases the vacuum is a constant, say, $|0> = \ 1$, as in the non-deformed case.

The following three operators

$$\tilde{J}^+_\alpha \ = \ \tilde{b}^2\tilde{a} - \{\alpha\}\tilde{b} \ ,$$

$$\tilde{J}^0_\alpha \ = \ \tilde{b}\tilde{a} - \hat{\alpha} \ , \qquad \text{(Q.8)}$$

$$\tilde{J}^- \;=\; \tilde{a},$$

where $\{\alpha\} = \frac{1-q^\alpha}{1-q}$ is so called q-number and $\hat{\alpha} \equiv \frac{\{\alpha\}\{\alpha+1\}}{\{2\alpha+2\}}$, are generators of the q-deformed or quantum sl_{2q}-algebra. The operators (Q.8) after multiplication by some factors, become

$$\tilde{j}^0 = \frac{q^{-\alpha}}{q+1}\frac{\{2\alpha+2\}}{\{\alpha+1\}}\tilde{J}^0_\alpha \ ,$$
$$\tilde{j}^\pm = q^{-\alpha/2}\tilde{J}^\pm_\alpha \ ,$$

and span the quantum algebra sl_{2q} with the standard commutation relations [6][6],

$$\begin{aligned} \tilde{j}^0\tilde{j}^+ \;-\; q\tilde{j}^+\tilde{j}^0 \;&=\; \tilde{j}^+ \ , \\ q^2\tilde{j}^+\tilde{j}^- \;-\; \tilde{j}^-\tilde{j}^+ \;&=\; -(q+1)\tilde{j}^0 \ , \\ q\tilde{j}^0\tilde{j}^- \;-\; \tilde{j}^-\tilde{j}^0 \;&=\; -\tilde{j}^- \ . \end{aligned} \tag{Q.9}$$

Comment. The algebra (Q.9) is known in literature as *the second Witten quantum deformation* of sl_2 in the classification of C. Zachos [8]).

In general, for the quantum sl_{2q} algebra there are no polynomial Casimir operators (see, for example, Zachos [8]). However, in the representation (Q.8) a relationship between generators analogous to the quadratic Casimir operator appears

$$q\tilde{J}^+_\alpha\tilde{J}^-_\alpha - \tilde{J}^0_\alpha\tilde{J}^0_\alpha + (\{\alpha+1\} - 2\hat{\alpha})\tilde{J}^0_\alpha = \hat{\alpha}(\hat{\alpha} - \{\alpha+1\}) \ .$$

If $\alpha = n$ is a non-negative integer, then (Q.8) possesses a finite-dimensional irreducible representation in the Fock space (cf.(A.1.6))

$$\mathcal{P}_n(\tilde{b}) \;=\; \langle 1, \tilde{b}, \tilde{b}^2, \ldots, \tilde{b}^n \rangle \ , \tag{Q.10}$$

of the dimension $\dim \mathcal{P}_n = (n+1)$.

References

[1] Y. F. Smirnov and A. V. Turbiner, "Lie-algebraic discretization of differential equations", *Modern Physics Letters* **A10**, 1795-1802 (1995), ERRATUM-*ibid* **A10**, 3139 (1995); "Hidden sl_2-algebra of finite-difference equations', Proceedings of IV Wigner Symposium, World Scientific, 1996, N.M. Atakishiyev, T.H. Seligman and K.B. Wolf (Eds.), pp. 435-440

[2] A.M. Perelomov, "Generalized coherent states and its applications", Nauka, 1987 (in Russian)

[3] A. González-Lopéz, N. Kamran and P.J. Olver, "Quasi-Exactly-Solvable Lie Algebras of the first order differential operators in Two Complex Variables", *J.Phys.***A24** (1991) 3995-4008; "Lie algebras of differential operators in two complex variables", *American J. Math.***114** (1992) 1163-1185

[4] L. Brink, A. Turbiner and N. Wyllard, "Hidden Algebras of the (super) Calogero and Sutherland models," Preprint ITP 97-05 and ICN-UNAM/97-02; hep-th/9705219

[5] M.A. Shifman and A.V. Turbiner, "Quantal problems with partial algebraization of the spectrum", *Comm. Math. Phys.* **126** (1989) 347-365

[6] O. Ogievetsky and A. Turbiner, " $sl(2,\mathbf{R})_q$ and quasi-exactly-solvable problems", Preprint CERN-TH: 6212/91 (1991) (unpublished)

[6]For discussion see [7] as well

[7] A. V. Turbiner, "On Polynomial Solutions of Differential Equations", *Journ. Math. Phys.* **33**, 3989-3994 (1992);
"Lie algebras and linear operators with invariant subspace," in *Lie algebras, cohomologies and new findings in quantum mechanics* (N. Kamran and P. J. Olver, eds.), AMS, vol. 160, pp. 263–310, 1994;
"Lie-algebras and Quasi-exactly-solvable Differential Equations", in *CRC Handbook of Lie Group Analysis of Differential Equations*, Vol.3: New Trends in Theoretical Developments and Computational Methods, Chapter 12, CRC Press (N. Ibragimov, ed.), pp. 331-366, 1995

[8] C. Zachos, "Elementary paradigms of quantum algebras", AMS *Contemporary Mathematics,* **134**, 351-377; J. Stasheff and M. Gerstenhaber (eds.), AMS, 1991